Bonn Research Presentation

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Compensated Compactness

Invariants of Hypersurface Complements

Ginzburg-Landau

Consider $B_1 \subseteq \mathbb{R}^2$ and suppose we had a smooth map $g : \partial B_1 \to S^1$ which we wanted to extend to some

$$u : \bar{B}_1 \to S^1$$
$$u|_{\partial B_1} = g.$$

If we had such an extension then

$$\begin{aligned} H : \partial B_1 \times [0,1] &\to S^1 \\ (x,t) &\mapsto u(tx) \end{aligned}$$

defines a homotopy from

$$\begin{aligned} H(x,0) &= u(0), \quad \text{a constant map, to} \\ H(x,1) &= u(x) = g(x). \end{aligned}$$

So, for such an extension to exist, we need $\deg(g) = 0$.

Ginzburg-Landau II

What if $\deg(g) \neq 0$? The idea here is to allow our u to take values in $\mathbb{C}$ instead of S^1, but require it to minimize the **Ginzburg-Landau energy**:

$$E_\kappa(u; B_1) := \int_{B_1} \left[\frac{1}{2}|\nabla u|^2 + \frac{\kappa^2}{4}(1 - |u|^2)^2 \right] d^n x$$

amongst $u \in W^{1,2}(B_1, \mathbb{C})$ with $u|_{\partial B_1} = g$.

As we take $\kappa \to \infty$, this energy functional penalizes u when it doesn't take values in S^1.

Ginzburg-Landau III

Example

We can study the Ginzburg-Landau functional when $B_1 \subseteq \mathbb{R}^n$ for $n \neq 2$ as well. If $n = 1$ we can consider

$$u : \mathbb{R} \to \mathbb{R}$$
$$u(x) = \tanh(\kappa x/\sqrt{2}).$$

Then indeed $-u''(x) = \kappa^2 u(x)(1 - u(x)^2)$. Furthermore we have

$$\int_{-1}^{1} \left[\tfrac{1}{2}|u'|^2 + \tfrac{\kappa^2}{4}(1-u^2)^2 \right] d^n x = \kappa \left(\tfrac{2}{\sqrt{2}} \tanh(\kappa/\sqrt{2}) - \tfrac{\sqrt{2}}{3} \tanh^3(\kappa/\sqrt{2}) \right)$$

which is bounded above by a constant multiple of κ as $\kappa \to \infty$.

Superconductivity

- In the case where u takes values in $\mathbb{C}$, then u can describe the wavefunction for a “fluid” of Cooper pairs in a superconductor.
- In this case it is typical to include a term in the energy proportional to the energy of the induced magnetic field.
- κ has physical meaning as the ratio of the London penetration depth to coherence length, and the $\kappa \to \infty$ approximation usually comes with the assumption that

$$E_\kappa(u_\kappa; B_1) \lesssim \log(\kappa).$$

This is since $\log(\kappa)$ is the characteristic energy of a radius κ^{-1} vortex:

$$u(x, y) = \frac{z}{|z|} = \frac{x + iy}{\sqrt{x^2 + y^2}} \text{ on } \{x \in \mathbb{R}^2 \ : \ \kappa^{-1} < |x| < 1\}$$

describing the region in which the magnetic field can penetrate the metal.

Separating Phase and Amplitude

In the Ginzburg-Landau equations, one often estimates the amplitude and the phase separately. Using path lifting, we can locally write

$$u = \rho e^{i\varphi} \quad \text{for } \rho, \varphi \text{ real valued}$$

we can compute:

$$|\nabla u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \varphi|^2.$$

A key technique in the modern theory of the Ginzburg-Landau equations is to write

$$\rho^2 |\nabla \varphi|^2 = \nabla \varphi \cdot (\rho^2 \nabla \varphi) = \nabla \varphi \cdot \mathrm{Im}(\bar{u} \nabla u)$$

and to notice that the Ginzburg-Landau equations imply

$$\mathrm{div}(\mathrm{Im}(\bar{u} \nabla u)) = 0.$$

As we will see: this "grad-curl" structure of the quantity $\rho^2 |\nabla \varphi|^2$ will give us improved control.

Hardy Space

- Unfortunately $L^1(\mathbb{R}^n)$ has no predual. If $f_j \in L^1(\mathbb{R}^n)$ is uniformly bounded, the best we can do it consider

$$|f_j| d^n x \in \mathrm{Meas}(\mathbb{R}^n) \cong (C_b^0(\mathbb{R}^n))^*.$$

- After passing to a subsequence we get $|f_j| d^n x \rightharpoonup \mu$ for some $\mu \in \mathrm{Meas}(\mathbb{R}^n)$. When is μ absolutely continuous with respect to Lebesgue measure?

Example

For $\phi \in C_c^\infty(B_1)$ with $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(x) d^n x = 1$ we can introduce $\phi_t(x) := t^{-n}\phi(x/t)$. Then $\|\phi_t\|_{L^1} = 1$ for all $t > 0$ and yet

$$\lim_{t \searrow 0} \phi_t = \delta_0.$$

Theorem

If $\phi \in L^1(\mathbb{R}^n)$ *then* $\sup_{r>0} |(\phi_r * f)(x)| \leq \|\phi\|_{L^1}(Mf)(x)$*.*

Sketch of Proof.

We will do the proof for ϕ a simple function $\phi(x) = \sum_j a_j \mathbb{1}_{B_{r_j}}(x)$ with $a_j > 0$ for all j. The general case is obtained from the fact that simple functions can be used to approximate any L^p function. We compute

$$
\begin{aligned}
(\phi * f)(x) &= \sum_j a_j (\mathbb{1}_{B_{r_j}} * f)(x) \\
&= \sum_j a_j \operatorname{Vol}(B_{r_j}) \left(\frac{1}{\operatorname{Vol}(B_{r_j})} \mathbb{1}_{B_{r_j}} * f \right)(x) \\
&\leq \left(\sum_j a_j \operatorname{Vol}(B_{r_j}) \right) (Mf)(x) = \|\phi\|_{L^1} Mf(x). \qquad \square
\end{aligned}
$$

Hardy Space II

- Pointwise a.e. convergence is determined by the *Hardy-Littlewood maximal operator*:

$$\begin{aligned}(Mf)(x) &:= \sup_{r>0} \frac{1}{\mathrm{Vol}(B_r)} (\mathbb{1}_{B_r} * |f|)(x) \\ &= \sup_{r>0} \frac{1}{\mathrm{Vol}(B_r(x))} \int_{B_r(x)} |f(y)| d^n y.\end{aligned}$$

- If we replace $K_r(x) := \frac{1}{\mathrm{Vol}(B_r)} \mathbb{1}_{B_r}(x)$ with a more general approximate identity $\phi_r(x) = \frac{1}{\mathrm{Vol}(B_r)} \phi(x/r)$ where $\phi \in C_c^\infty(B_1)$ has $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(x) d^n x = 1$, and if we take absolute value after convolving instead of before convolving we obtain different sort of maximal function

$$(M_\phi f)(x) := \sup_{r>0} |(\phi_r * f)(x)| = \sup_{r>0} \left| \frac{1}{\mathrm{Vol}(B_r)} \int_{B_r(x)} \phi_r(x-y)) f(y) d^n y \right|.$$

BMO

Our earlier theorem showed that

$$M_\phi f \leq \|\phi\|_{L^1}(Mf) \text{ pointwise.}$$

One problem related to bounding f pointwise is that L^p functions need not be bounded. A more relevant quantity is the **mean oscillation** of $f \in L^1_{loc}(\mathbb{R}^n)$ on a cube Q, given by:

$$\operatorname{osc}(f, Q) := \frac{1}{\operatorname{Vol}(Q)} \int_Q |f(y) - \bar{f}_Q| d^n y$$

where $\bar{f}_Q := \frac{1}{\operatorname{Vol}(Q)} \int_Q f(y) d^n y$. We say that f is in **BMO** if and only if the following maximal function

$$(M^\# f)(x) := \sup_{Q \ni x} \operatorname{osc}(f, Q) \text{ is in } L^\infty(\mathbb{R}^n)$$

where $Q \ni x$ ranges over all dyadic cubes containing x parallel to the coordinate axis.

BMO II

- The BMO norm (technically constant functions have norm zero) is $\|M^{\#} f\|_{L^\infty}$.
- The idea for the mean oscillation arises from the Calderón-Zygmund decomposition $f = g + b$

$$\frac{1}{|Q|}\int_Q |f - f_Q| \sim \frac{1}{|Q|}\int_Q |b|, \qquad g = \frac{1}{|Q|}\int_Q f$$

$$\lambda < \frac{1}{|Q|}\int_Q |f| \leq 2^n \lambda, \qquad \frac{1}{|Q|}\int_Q |f| \simeq \lambda.$$

- The oscillation maximal function is also pointwise controlled by the Hardy-Littlewood maximal function

$$M^{\#} f \lesssim Mf.$$

- One can show that Calderón-Zygmund operators are bounded from L^∞ to BMO.

BMO III

Example

The Hilbert transform

$$(Hu)(x) := \frac{1}{\pi} \text{ p.v. } \int_{-\infty}^{\infty} \frac{u(y)}{x-y} dy$$

is bounded from $L^\infty(\mathbb{R})$ to $\mathrm{BMO}(\mathbb{R})$. The point of Hu is:

if $u(x) = \mathrm{Re}(f(x+i0))$ for $f(z)$ holomorphic when $\mathrm{Im}(z) > 0$ then

$$\mathrm{Im}(f(x+i0)) = Hu(x).$$

Testing this on the L^∞ function $u(x) := x/|x|$ we get

$$(Hu)(x) = \frac{2}{\pi} \log |x| + C.$$

BMO IV

Example

(continued) Since $\log|x|$ is unbounded we see that

$$L^\infty(\mathbb{R}^n) \subsetneq \mathrm{BMO}(\mathbb{R}^n).$$

Furthermore, the function $f(z) = \frac{2}{\pi} i \log(iz)$ is holomorphic when $\mathrm{Im}(z) > 0$ and we can compute:

$$f(z) = \frac{2i}{\pi} \log(x^2 + y^2)^{1/2} + \frac{2}{\pi} \arctan(x/y) \quad \text{and}$$
$$\lim_{y \to 0} f(z) = i\frac{2}{\pi} \log|x| + x/|x|.$$

Hardy Space III

Example

Here we give more evidence that it is unreasonable to assue one has control on Mf in general. Suppose $E \subseteq B_R$ is measurable and $f(x) = \mathbb{1}_E(x)$. Then

$$\sup_{r>0} \frac{1}{\mathrm{Vol}(B_r(x))} \int_{B_r(x)} |f(y)| d^n y \geq \frac{\mathrm{Vol}(E \cap B_R(x))}{\mathrm{Vol}(B_R(x))} \geq \frac{C_n \mathrm{Vol}(E)}{(R+|x|)^n}$$

so that

$$\int_{\mathbb{R}^n} (Mf)(x) d^n x \geq C_n C_{n-1} \mathrm{Vol}(E) \int_0^\infty \frac{r^{n-1}}{(R+r)^n} dr = \infty.$$

More generally, one can show that if $g \in L^1(\mathbb{R}^n)$ and $Mg \in L^1(\mathbb{R}^n)$ then $g = 0$. Thus we can't hope for uniform L^1 bounds on the maximal operator.

Hardy Space IV

Definition

Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ is defined to be the set of all $f \in L^1(\mathbb{R}^n)$ such that $M_\phi f \in L^1(\mathbb{R}^n)$. This is a Banach space with norm

$$\|f\|_{\mathcal{H}^1} := \|f\|_{L^1} + \|M_\phi f\|_{L^1}.$$

- Recall that in M_ϕ we take an absolute value only after convolving with ϕ_r!
- The main point of Hardy space comes from its relationship with the spaces VMO and BMO.

VMO

Definition

$$\mathrm{VMO}(\mathbb{R}^n) := \left\{ f \in \mathrm{BMO}(\mathbb{R}^n) \ : \ \lim_{r \searrow 0} \mathrm{osc}(f, B_r(x)) = 0 \text{ a.e.} \right\}.$$

One of the most important theorems concerning Hardy space is the Fefferman-Stein theorem below.

Theorem

$\mathcal{H}^1(\mathbb{R}^n)$ is the dual of $\mathrm{VMO}(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ is the dual of Hardy space.

As a consequence, sequences $f_n \in \mathcal{H}^1(\mathbb{R}^n)$ with $\|f_n\|_{\mathcal{H}^1}$ uniformly bounded have weakly converging sequences that converge to actual functions, not measures!

Compensated Compactness

Theorem

Let $X \in L^2(\mathbb{R}^n, \mathbb{R}^n)$ and $\psi \in L^2_{loc}(\mathbb{R}^n)$ be such that $\nabla\psi \in L^2(\mathbb{R}^n, \mathbb{R}^n)$ and $div(X) = 0$. Then $\langle \nabla\psi, X\rangle \in \mathcal{H}^1(\mathbb{R}^n)$ with

$$\|\langle \nabla\psi, X\rangle\|_{\mathcal{H}^1} \lesssim \|\nabla\phi\|_{L^2}\|X\|_{L^2}.$$

Proof.

By Hölder's inequality we have

$$\|\langle \nabla\psi, X\rangle\|_{L^1} \leq \|\nabla\psi\|_{L^2}\|X\|_{L^2}$$

so it suffices to bound

$$\sup_{t>0} |(\phi_t * \langle \nabla\psi, X\rangle)(x)| \quad \text{in} \quad L^1(\mathbb{R}^n).$$

Compensated Compactness II

We apply the divergence theorem, using that $\operatorname{div}(X) = 0$,

$$
\begin{aligned}
|(\phi_t * \langle \nabla\psi, X\rangle)(x)| &= \Big| \int_{\mathbb{R}^n} \langle \nabla\psi, X\rangle(y) t^{-n} \phi((y-x)/t) d^n y \Big| \\
&= \Big| \int_{\mathbb{R}^n} \frac{(\psi(y) - \bar{\psi}_{x,t})}{t} t^{-n} X \cdot \nabla\phi((y-x)/t) d^n y \Big|
\end{aligned}
$$

where

$$
\bar{\psi}_{x,t} := \frac{1}{\operatorname{Vol}(B_t(x))} \int_{B_t(x)} \psi(z) d^n z.
$$

Using that the support of $(\nabla\phi)((y-x)/t)$ as a function of y is contained in $B_t(x)$ we get an upper bound:

$$
\leq \|\nabla\phi\|_{L^\infty}\; t^{-n} \int_{B_t(x)} \left| \frac{\psi(y) - \bar{\psi}_{x,t}}{t} \right| |X| d^n y.
$$

Compensated Compactness III

Choose $1 \leq a < 2$ and $1 \leq b < 2$ such that $\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{n}$. Since $\frac{1}{a} - \frac{1}{n} = 1/(an/(n-a))$ we can use Hölder to get

$$\leq \|\nabla\phi\|_{L^\infty} \left(t^{-n} \int_{B_t(x)} \left| \frac{\psi(y) - \bar{\psi}_{x,t}}{t} \right|^{an/(n-a)} d^n y \right)^{(a-p)/an} \left(t^{-n} \int_{B_t(x)} |X|^b d^n y \right)^{1/b}$$

and bound the first factor via the Gagliardo-Nirenberg-Sobolev inequality

$$\left(t^{-n} \int_{B_t(x)} \left| \frac{\psi(y) - \bar{\psi}_{x,t}}{t} \right|^{an/(n-a)} d^n y \right)^{(n-a)/na} \leq C_n \left(t^{-n} \int_{B_t(x)} |\nabla\psi|^a d^n y \right)^{1/a}$$

Conpensated Compactness IV

Recalling the definition of the Hardy-Littlewood maximal operator, we arrive at:

$$\sup_{t>0} |(\phi_t * \langle \nabla\psi, X\rangle)(x)| \lesssim \left(M|\nabla\psi|^a(x)\right)^{1/a} \left(M|X|^b(x)\right)^{1/b}.$$

Since $a < 2$ and $b < 2$ we have $|\nabla\psi|^a \in L^{2/a}$ and $|X|^b \in L^{2/b}$ with $2/a,\ 2/b > 1$. Since the maximal operator is bounded on $L^{2/a}$ and $L^{2/b}$ we can compute using Hölder to get:

$$\begin{aligned}
\left\| \sup_{t>0} |(\phi_t * \langle \nabla\psi, X\rangle(x)| \right\|_{L^1(dx)} &\lesssim \left\| (M|\nabla\psi|^a(x))^{1/a} \right\|_{L^2(dx)} \left\| (M|X|^b(x))^{1/b} \right\|_{L^2(dx)} \\
&= \left\| M|\nabla\psi|^a \right\|_{L^{2/a}}^{1/a} \left\| M|X|^b \right\|_{L^{2/b}}^{1/b} \\
&\lesssim \left\| |\nabla\psi|^a \right\|_{L^{2/a}}^{1/a} \left\| |X|^b \right\|_{L^{2,b}}^{1/b} \\
&= \|\nabla\psi\|_{L^2} \|X\|_{L^2}
\end{aligned}$$

as desired. $\square$

Remarks

- If we have a sequence ψ_k, X_k and uniform L^1 bounds on $\langle \nabla \psi_k, X_k \rangle$ then we can at most guarantee that the limit will be a measure $\mu \in \mathrm{Meas}(\mathbb{R}^n)$ after passing to a subsequence.
- However, if $\langle \nabla \psi_k, X_k \rangle$ is uniformly bounded in Hardy space then we can guarantee that the limit is a function since $\mathcal{H}^1(\mathbb{R}^n) = \mathrm{VMO}(\mathbb{R}^n)^*$.
- Compensated compactness says:

  We can compensate the deficiencies of L^1 bounds if ψ, X solve an equation.

Applications

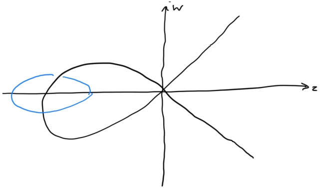
- For solutions u_1, u_2 to the minimal surface equations on $\Omega \subseteq \mathbb{R}^n$ we have
$$\operatorname{div}(A\nabla(u_2 - u_1)) = 0$$
for A a matrix with eigenvalues between μ and μ^{-1}, $\mu > 0$ depending only on upper bounds for $|\nabla u_1|$ and $|\nabla u_2|$. Does $A\nabla(u_2 - u_1)$ satisfy a compensated compactness?
- $\Sigma \subseteq \mathbb{R}^n$ is minimal if and only if the coordinate functions x^i on $\mathbb{R}^n$ are harmonic on Σ. If $\vec{e}_j$ is the j'th standard basis vector, then does $x^i\vec{e}_j - x^j\vec{e}_i$ satisfy a compensated compactness? For a minimal graph u in $\mathbb{R}^3$, the Poincaré inequality
$$\int_{\mathrm{Graph}(u)} |A|^2 f^2 \leq C \int_{\mathrm{Graph}(u)} |\nabla_{\mathrm{Graph}(u)} f|^2$$
for Lipschitz functions f could substitute for the Sobolev inequality in the proof.

The Linking Number Homomorphism

- Our goal is the study the complement $\mathbb{C}^2 \setminus C$ where $C = Z(f)$ is the zero set of $f \in \mathbb{C}[z, w]$ of degree m.
- The first step is to make use of the homomorphism

  $$f_* : \pi_1(\mathbb{C}^2 \setminus C) \to \pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z} \text{ (via deformation retraction)}$$

  called the **linking number homomorphism**. This roughly measures how many times loops in $\mathbb{C}^2 \setminus C$ wrap around C.

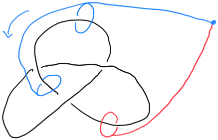


$$f(z, w) = w^2 - z^2(z + 1)$$

Example

The trefoil knot K has Wirtinger presentation $\langle x, y | x^2 = y^3 \rangle$. In the abelianization where $xy = yx$ we can compute:

$$(xy^{-1})^2 = x^2 y^{-2} = y \quad \text{and}$$
$$(xy^{-1})^3 = x^3 y^{-3} = x. \quad \text{So } H_1(\mathbb{R}^3 \setminus K) \cong \mathbb{Z}\,.$$



Non homotopic loops in $\mathbb{R}^3 \setminus K$:

The Linking Number Homomorphism II

- Since $\mathbb{Z}$ is torsion-free abelian and the abelianization of $\pi_1(\mathbb{C}^2 \setminus C)$ is $H_1(\mathbb{C}^2 \setminus C)$, f_* induces a homomorphism

$$\bar{f}_* : \Gamma_0 := H_1(\mathbb{C}^2 \setminus C)/\{\text{torsion}\} \to \mathbb{Z}\,.$$

- As $\pi_1(\mathbb{C} \setminus C)$ is finitely generated, Γ_0 is free abelian by the classification of finitely generated abelian groups.
- A choice of $[\gamma] \in \Gamma_0$ such that $\bar{f}_*([\gamma]) = 1$ is called a **meridian**. This defines a homomorphism

$$\begin{aligned} \phi : \mathbb{Z} &\to \Gamma_0 \\ 1 &\mapsto [\gamma] \end{aligned}$$

which splits $\bar{f}_*$, i.e. $\bar{f}_* \circ \phi = \mathrm{id}_{\mathbb{Z}}$.

Meridians

What is the rank of the free abelian group Γ_0? If we factor $f(z,w) = f_1(z,w)\cdots f_s(z,w)$ into a product of irreducibles in $\mathbb{C}[z,w]$ then we have

$$C = \bigcup_{i=1}^{s} Z(f_i).$$

The zero sets $Z(f_i)$ are called the **irreducible components** of C and the rank of Γ_0 is the number s of irreducible components. We illustrate this with an example.

Example

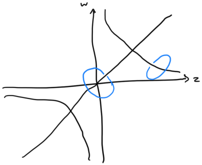
Consider the degree 3 reducible polynomial $f(w,z) = (wz-1)(w-z) \in \mathbb{C}[z,w]$. The intersection of its zero set with $\mathbb{R}^2$ is pictured below.

Meridians II

Example

(continued) Two non-homotopic loops are as follows:

$$\gamma_1(\theta) := ((1/2)e^{2\pi i\theta}, -(1/2)e^{2\pi i\theta})$$
$$\gamma_2(\theta) := ((1/2)e^{2\pi i\theta}, (1/2)e^{2\pi i\theta}) + (4, 1/4)$$



Invariants for Plane Curve Complements

- One can always choose a basis of meridians $[\gamma_1], \dots, [\gamma_s] \in \Gamma_0$. Fixing $\phi : \mathbb{Z} \to \Gamma_0$ to be the splitting corresponding to γ_1 we see that $\bar{f}_*([\gamma_i] \cdot \phi(\bar{f}_*[\gamma_1])^{-1}) = 1 - 1 = 0$ and so if $t_j := [\gamma_j]$ then

$$\begin{aligned} \Gamma_0 &\xrightarrow{\cong} \mathbb{Z} \oplus \ker(\bar{f}_*) \\ g &\mapsto (\bar{f}_*(g), g \cdot \phi(\bar{f}_*(g))^{-1}) \end{aligned}$$

  is a group isomorphism where $t_j/t_1 \mapsto (0, t_j/t_1)$ for $j \neq 1$ handles $\ker \bar{f}_*$ and $t_1 \mapsto (1,1)$ deals with $\mathbb{Z}$.
- In this way, we have isomorphisms of group rings

$$\mathbb{Z}[\Gamma_0] \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$$
$$\mathbb{Z}[\ker(\bar{f}_*)] \cong \mathbb{Z}\left[\left(\frac{t_2}{t_1}\right)^{\pm 1}, \dots, \left(\frac{t_s}{t_1}\right)^{\pm 1}\right]$$

Invariants for Plane Curve Complements II

- We will be primarily concerned with an invariant called $\delta_0(C)$ and its relationship to the classical Alexander polynomial.
- To be an invariant means that if C and C' are homotopic curves then $\delta_0(C) = \delta_0(C')$.
- Following our earlier notation we let $S_0 := \mathbb{Z}[\ker(\bar{f}_*)] \setminus \{0\}$ and

$$\mathbb{K}_0 := \mathrm{Frac}\left(\mathbb{Z}\left[\left(\frac{t_2}{t_1}\right)^{\pm 1}, \ldots, \left(\frac{t_s}{t_1}\right)^{\pm 1}\right]\right)$$

$$R_0 := (\mathbb{Z}[\Gamma_0])S_0^{-1} = \mathbb{K}_0[t_1^{\pm 1}].$$

Definition

We define

$$\delta_0(C) := \dim_{\mathbb{K}_0} H_1(\mathbb{C}^2 \setminus C; R_0).$$

Invariants III: Alexander Polynomials

- Recalling that $G = \pi_1(\mathbb{C}^2 \setminus C, c_0)$ and Γ_0 was the maximal torsion-free abelian quotient of G, the **Alexander module of** G is $H_1(\mathbb{C}^2 \setminus C, c_0; \mathbb{Z}[\Gamma_0])$.
- As a finitely generated module over the UFD $\mathbb{Z}[\Gamma_0]$, this has a finite presentation.

$$(\mathbb{Z}[\Gamma_0])^q \xrightarrow{A} (\mathbb{Z}[\Gamma_0])^p \to H_1(\mathbb{C}^2 \setminus C, c_0; \mathbb{Z}[\Gamma_0])$$

  For each collection of $p-1$ rows and $p-1$ columns of the matrix A, we can take a determinant to obtain an element of $\mathbb{Z}[\Gamma_0]$. We let $E_1 \trianglelefteq \mathbb{Z}[\Gamma_0]$ denote the ideal generated by these determinants.
- The **Alexander polynomial** of C is then:

$$\Delta_C := \gcd(E_1) \in \mathbb{Z}[\Gamma_0] = \mathbb{Z}[t_1^{\pm 1}, ..., t_s^{\pm 1}].$$

Invariants V

Theorem

Let $C \subseteq \mathbb{C}^2$ be a curve of degree $m \geq 2$ such that $\delta_0(C) < \infty$. Then

$$\delta_0(C) = \deg(\Delta_C).$$

Proof.

We begin by choosing a set of generators $f_1, \ldots, f_r$ for E_1, the ideal of $p-1$ minors of the matrix defining the presentation of $H_1(\mathbb{C}^2 \setminus C, c_0; \mathbb{Z}[\Gamma_0])$. In this way

$$\Delta_C = \gcd(f_1, \ldots, f_r) \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}].$$

Let E_1' denote the ideal of $p'-1$ minors of the matrix defining the analogous presentation of $H_1(\mathbb{C}^2 \setminus C, c_0; R_0)$ and notice that since R_0 is a localization of $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$ it follows from properties of localization that $f_1, \ldots, f_r$ also generate the ideal E_1'.

Invariants VI

So $E_1' = (f_1, \ldots, f_r) \trianglelefteq R_0 = \mathbb{K}_0[t_1^{\pm 1}]$ and, since $\mathbb{K}_0[t_1^{\pm 1}]$ is a PID, there exists $p \in \mathbb{K}_0[t_1^{\pm 1}]$ such that $E_1' = (f_1, \ldots, f_r) = (p)$ and so the long exact sequence in homology for the pair $(\mathbb{C}^2 \setminus C, c_0)$ yields:

$$\delta_0(C) = \deg_{\mathbb{K}_0[t_1^{\pm 1}]}(p).$$

So it suffices to show that $p = a \cdot \Delta_C$ for some $a \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$ which is a unit in $\mathbb{K}_0[t_1^{\pm 1}]$. However Δ_C is the gcd of $(f_1, \ldots, f_r)$ in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$ and is therefore a common divisor in $\mathbb{K}_0[t_1^{\pm 1}]$ while p is the gcd in $\mathbb{K}_0[t_1^{\pm 1}]$ so p divides Δ_C in $\mathbb{K}_0[t_1^{\pm 1}]$, say $ap = \Delta_C$. The fact that a is a unit in $\mathbb{K}_0[t_1^{\pm 1}]$ now arises from the fact than any prime element in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$ is either prime in $\mathbb{K}_0[t_1^{\pm 1}]$ or a unit in $\mathbb{K}_0[t_1^{\pm 1}]$. $\square$